MATH GU 4156 **Advanced Probability Theory** Columbia University (Fall 2023)

Lecture 6: Martingales Convergence

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6.1 Supermartingale Convergence

We posit now the following intuition:

Supermartingales are the probabilistic analogues of decreasing functions.

If we think of this aphorism at face value, we are led to conclude that supermartingales bounded from below must converge.

A spectacular result of Doob turns this intuition into mathematics, by showing that "bounded from below" means here

$$K := \sup_{m \in \mathbb{N}_0} \mathbb{E}(X_n^-) < \infty. \tag{6.1}$$

Theorem 6.1 (Doob Supermatingale Convergence) *For every supermartingale $\mathcal{X} = (X_n)_{n \in N_0}$ that satisfies the above condition (6.1), the limit*

$$X_\infty = \lim_{n \to \infty} X_n$$

exists $\mathbb{P}$-a.e., and is integrable: $\mathbb{E}|X_\infty| < \infty$.

In particular, every nonnegative supermartingale converges. The proof uses the following, ingenuous inequality.

Lemma 6.2 (Doob's Upcrossing Inequality) *In the above context,*

$$\mathbb{E}\left[U_n(a,b;\mathcal{X})\right] \le \frac{\mathbb{E}(X_n - a)^-}{b-a} \le \frac{|a| + \mathbb{E}(X_n)^-}{b-1};\ a < b,\ n \in \mathbb{N}. \tag{6.2}$$

Here, $U_n(a,b;\mathcal{X})$ is the total number of upcrossings, from below $a \in \mathbb{R}$ to above $b \in \mathbb{R}$, $b > a$, that the sequence $\mathcal{X}$ has **completed** by time $t = n$.

Here, we introduce the stopping times

$$\tau_0 \equiv 0,\ \tau_1 := \min\{k : X_k \le a\}\ \tau_2 := \min\{k > \tau_1 : X_k \ge b\}$$

and inductively

$$\tau_{2m-1} := \min\{k > \tau_{2m} : X_k \le a\}\ \tau_{2m} := \min\{k > \tau_{2m-1} : X_k \ge b\},$$

as well as

$$U_n(a,b;\mathcal{X}) = \begin{cases} \max\{m \in \mathbb{N} : \tau_{2m} \le n\}; & \text{if } \tau_2 \le n \\ 0; & \text{if } \tau_2 > n. \end{cases}$$

For instance, in Figure 6.1, $U_n(a,b;\mathcal{X}) = 2$.

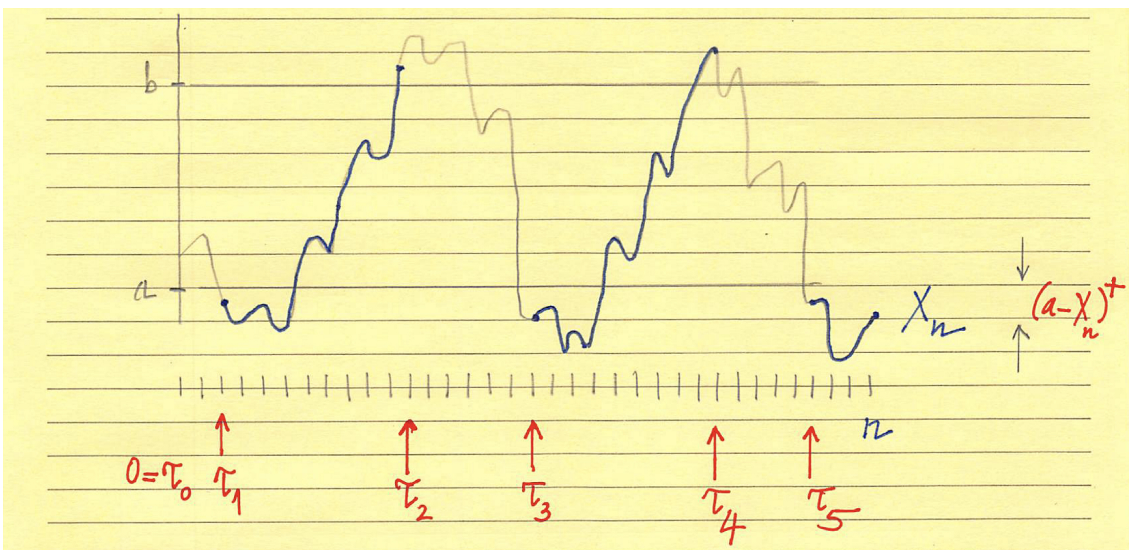

Figure 6.1: Sample path of $\mathcal{X}$.

Quite obviously $n \mapsto U_n(a,b;\mathcal{X})$ is increasing, so

$$U_\infty(a,b;\mathcal{X} = \lim_{n\to\infty} \uparrow U_n(a,b;\mathcal{X})$$

exists in $\mathbb{N} \cup \{\infty\}$: the total number of times $\mathcal{X}$ crosses from below a to above b, during its lifetime.

Needless to say, a similar inequality holds for submartingales, if you replace upcrossings by downcrossings, and negative parts by positive parts.

Proof: [Lemma 6.2] Think of X_n as the prove of an asset (oil, gold,...) on day $t = n$; and of yourself as investor. You set yourself to thresholds, a (low) and b (high), and adopt the following strategy: You buy one share on the first day the price falls to or below the threshold a; and keep buying one share a day, for as long as the price stay below the level b. Once this upper level is reached or exceeded, you exit; and you remain on the side-lines up until the next deop to the level a or below; and so on. Formally, you strategy is

$$\theta_j = \begin{cases} 1; \text{ if } \tau_m \leq \tau_{m+1}, \text{ for some odd m} \\ 0; \text{ if } \tau_m \leq \tau_{m+1}, \text{ for some even m} \end{cases},$$

and satisfies

$$\{\theta_j = 1\} = \cup_{k\in\mathbb{N}}\{\tau_{2k-1} < j \leq \tau_{2k}\} = \cup_{k\in\mathbb{N}}\left(\underbrace{\{\tau_{2k-1} < j\}}_{\in\mathcal{F}_{j-1}} - \underbrace{\{\tau_{2k-1} < j\}}_{\in\mathcal{F}_{j-1}}\right) \in \mathcal{F}_{j-1}.$$

This is because all the τ's are stopping times. As a consequence $\Theta = \{\theta_j\}_{j\in\mathbb{N}}$ is nonnegative, predictable.

What is the P&L ("profits and losses", "value", ...) resulting form this strategy? Quite obviously,

$$Y_0 = 0; \;\; Y_N = (\Theta \cdot \mathcal{X})_n = \sum_{j=1}^{n} \theta_j(X - j - X_{j-1}) \;\; (n \in \mathbb{N})$$

(the transform of the supermartingale $\mathcal{X}$ by the nonnegative, predictable Θ, thus a supermartingale itself), as well as

$$\begin{aligned} Y_n \geq & U_n(a, b; \mathcal{X})(b-a) \rightarrow \text{you make at least this amount on each upcrossing that gets completed.} \\ & -(a - X_n)^+ \qquad \rightarrow \text{the most you can lose on an upcrossing still in progress on day } t = n. \end{aligned}$$

The supermartingale property gives now

$$0 = \mathbb{E}Y_0 \geq \mathbb{E}Y_n \geq (b-a)\mathbb{E}[U_n(a, b; \mathcal{X})] = \mathbb{E}(X_n - a)^+.$$

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Proof: **[Theorem 6.1]** Letting $n \to \infty$ in the inequality, we get $\mathbb{E}[U_\infty(a, b; \mathcal{X})] \leq \frac{|a|+K}{b-a}$, by Monotone Convergence. In particular,

$$\mathbb{P}(U_\infty(a, b; \mathcal{X}) = \infty) = 0.$$

Now the event

$$\Lambda := \{\mathcal{X} \text{ does not converge in } [-\infty, \infty]\}$$

can be expressed as a countable union

$$\Lambda = \{\liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n\} = \cup_{a<b,(a,b)\in\mathbb{Q}^2}\Lambda_{a,b}$$

$$\Lambda_{a,b} := \{\liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n\} \subseteq \{U_\infty(a,b;\mathcal{X}) = \infty\}.$$

Thus $\mathbb{P}(\Lambda_{a,b}) = 0$ for each pair (a,b) as above, and $\mathbb{P}(\Lambda) = 0$. In other words, $X_\infty = \lim_{n\to\infty} X_n$ exists $\mathbb{P}$-a.e.

Now $|X_n| = X_n^+ + X_n^- = X_n + 2X_n^-$, therefore $\mathbb{E}|X_n| \le \mathbb{E}(X_0) + 2K =: L < \infty$; and by Fatou,

$$\mathbb{E}|X_\infty| = \mathbb{E}(\lim_n |X_n|) \le \liminf_{n\to\infty} \mathbb{E}|X_n| \le L < \infty.$$

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Closure: We say that $X_0, X_1, ..., X_\infty$ is an $\mathbb{F}$-martingale (resp, supermartingale, submartingale) with last element, if

$$\mathbb{E}(X_m|\mathcal{F}_n) = X_n \quad (\text{resp}, \ \le, \ge)$$

holds for every $n \in N_0$ and $m = n+1, ..., \infty$. Here we require X_∞ to be $\mathcal{F}_\infty := \sigma(\cup_{k\in\mathbb{N}}\mathcal{F}_k)$-measurable.

For instance: every nonnegative supermartingale can be extended to a supermartingale with last element $X_\infty = 0$.

Also, a Lévy martingale $X_n = \mathbb{E}(\xi|\mathcal{F}_n), \ n \in \mathbb{N}_0$, can thus be extended, with $X_\infty := \mathbb{E}(\xi|\mathcal{F}_\infty)$.

Indeed, by the tower property, we have

$$\mathbb{E}(X_\infty|\mathcal{F}_n) = \mathbb{E}[\mathbb{E}(\xi|\mathcal{F}_\infty)|\mathcal{F}_n] = \mathbb{E}(\xi|\mathcal{F}_n) = X_n.$$

Proposition 6.3 *For every nonnegative supermartingale $\mathcal{X} = (X_n)_{n\in\mathbb{N}_0}$ the limit $X_\infty = \lim_{n\to\infty} X_n$ exists, is real-valued, and the extended $X_0, X_1, X_2, ..., X_\infty$ is a supermartingale with last element.*

Proof: Doob supermartingale convergence gives the existence of X_∞, and the rest is Fatou:

$$X_n \ge \liminf_m \mathbb{E}(X_m|\mathcal{F}_n) \ge \mathbb{E}(\liminf_m X_m|\mathcal{F}_n) = \mathbb{E}(X_\infty|\mathcal{F}_n), \ \forall n \in \mathbb{N}_0.$$

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6.2 Jean Ville's Theorem

We should be remiss, if we failed to mention at this point a striking characterization of events of zero probability, due to Jean Ville (1939):

Theorem 6.4 *An event $A \in \mathcal{F}$ has $\mathbb{P}(A) = 0$ if, and only if, there exists a nonnegative martingale $\{M_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} M_n(\omega) = \infty$ valid for every $\omega \in \Omega$.*

I learned about this result only recently from my student Johannes Ruf who, with collaborators, has proved a very interesting extension of this result in the context of an entire family of probability measures (arXiv, April 2022).

Quite a bit more generally, given any event $E \in \mathcal{F}$, consider the collection M_E of nonnegative martingale $(M_n)_{n\in\mathbb{N}_0}$ with $\liminf_{n\to\infty} M_n \geq \mathbb{I}_E$ (i.e., which eventually reach or exceed the level 1, if E occurs). Then

$$\mathbb{P}(E) = \inf_{\{M_n\}_{n\in\mathbb{N}_0} \in M_E} (M + 0).$$